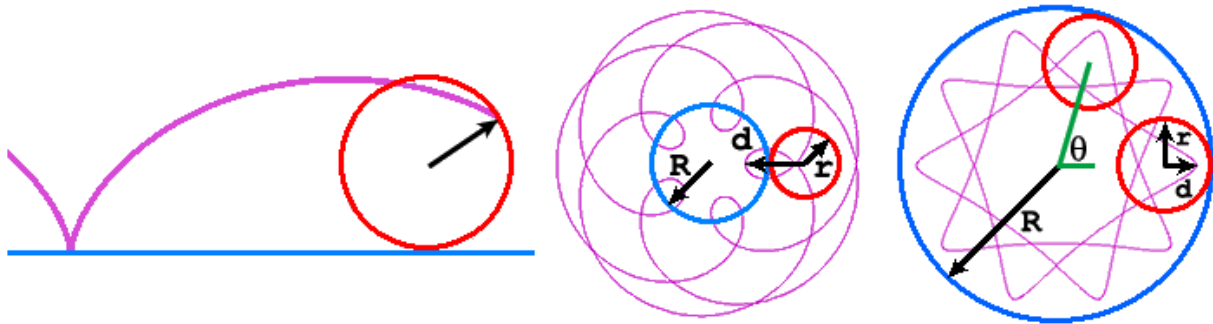


Roulettes and Related Curves

Cycloids and trochoids are subsets of roulettes. A roulette is a **curve** described by a generating point attached to a **moving curve** as that **curve** rolls along a **second curve** that is fixed. Either curve can be linear, open or closed. If the **rolling curve** is a circle and the **fixed curve** is a line, the roulette is a **trochoid**. If the generating point lies *on* the circle, the roulette is a **cycloid**. If the **fixed curve** is a circle, the roulette is an **epitrochoid** if the **rolling circle** is *outside* the **fixed circle**, and it is a **hypotrochoid** if the **rolling circle** is *inside* the **fixed circle**. **Epicycloids** and **hypocycloids** are subsets in which the generating point lies *on* the **rolling circle**.



Toys such as the popular Spirograph use plastic circles with gear teeth to ensure precise rolling. They are limited to trochoids in which the generating points (*i.e.* pens) are located in small holes inside the solid rolling circle. In abstract mathematics, a generating point can be outside the rolling circle, and it even is possible for the rolling circle of a hypotrochoid to have a larger diameter than that of the fixed circle. However, the Spirograph easily can use elliptical, triangular, rectangular or otherwise distorted shapes which are difficult to program.

The radius of the fixed circle is designated **R** and that of the moving circle is designated **r**. The distance of the generating point from the center of the rolling circle is designated **d**. As described above, cycloids are trochoids where **d** = **r**. Additionally, **d** can be zero, less than **r** or greater than **r**, and **r** can be greater than **R**. The **angle** made by the center of the moving circle and the **x** axis is designated **θ**. A curve is closed or completed (returning to its starting point) when the cumulative circumference of successive **rotations** of the moving circle exactly equals a multiple of the circumference of the fixed circle (**revolutions**). When **R** / **r** (the ratio of the radii and therefore the circumferences) is reduced to lowest integral terms, **R** = the number of rotations and **r** = the number of revolutions.

In the following parametric equations, the first term locates the center of the moving circle and the second locates the generating point using the rotational angle and the value of **d**:

epitrochoid

$$x = (R + r) \cos \theta - d \cos \left(\frac{R + r}{r} \theta \right)$$

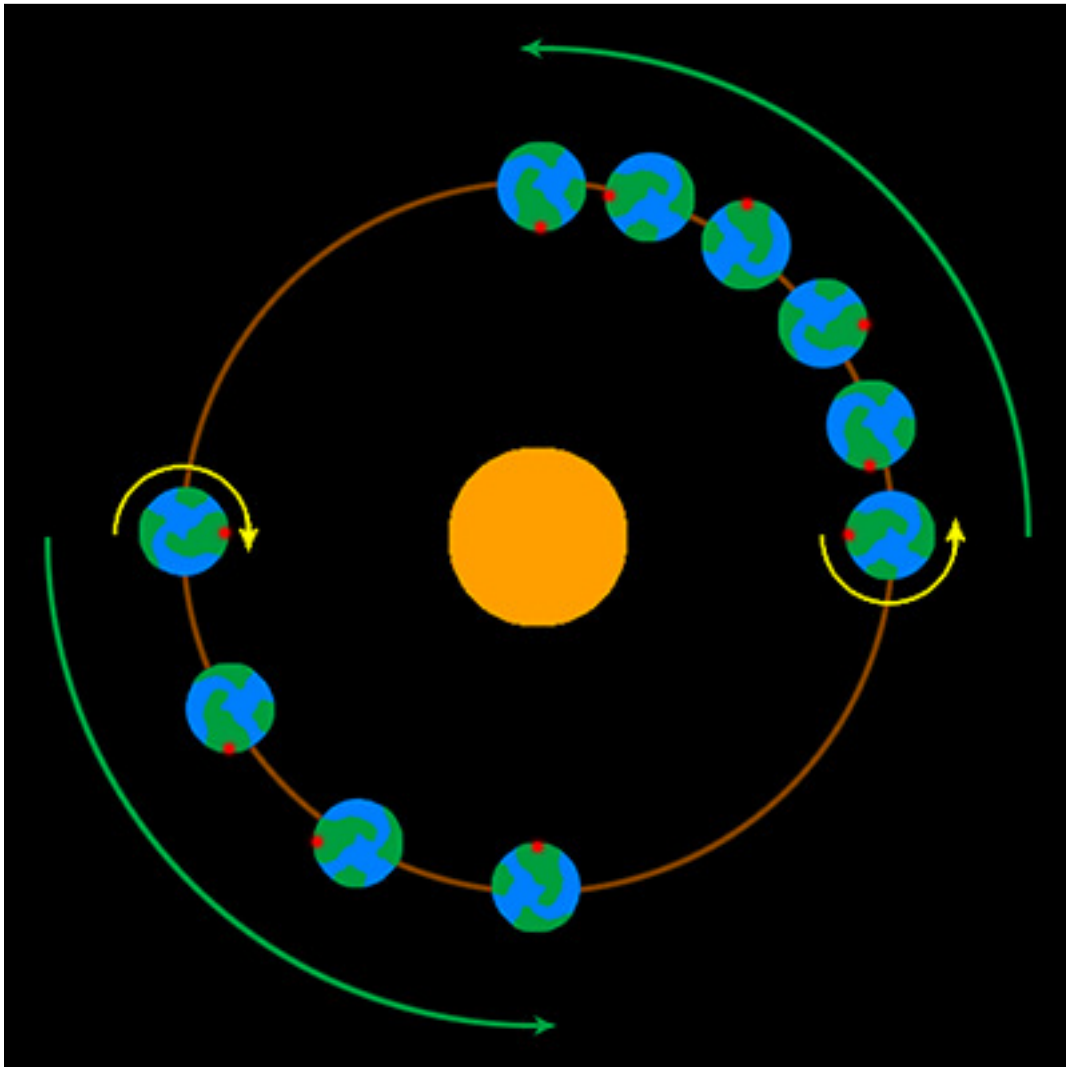
$$y = (R + r) \sin \theta - d \sin \left(\frac{R + r}{r} \theta \right)$$

hypotrochoid

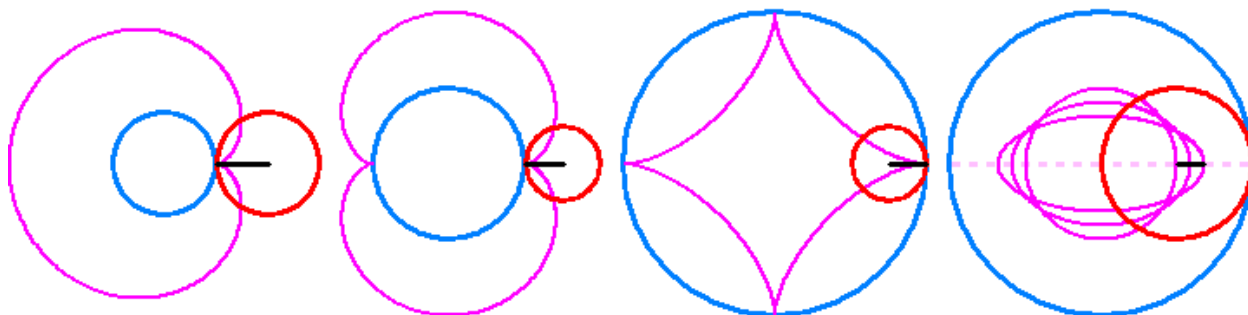
$$x = (R - r) \cos \theta + d \cos \left(\frac{R - r}{r} \theta \right)$$

$$y = (R - r) \sin \theta - d \sin \left(\frac{R - r}{r} \theta \right)$$

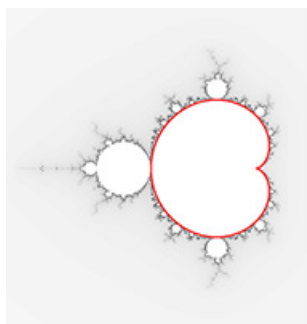
The number of rotations of the moving circle can be expressed in relation to either the x axis or the parametric angle θ . This concept can be visualized with an analogy. The diagram below shows a hypothetical planet in a circular orbit around a star. For each orbital revolution ("year"), the planet makes 4 rotations ("days"). At the beginning and end of each day, someone standing at the red dot will be facing the star directly. Note that if the planet rotates in a counter-clockwise direction, it makes 1 rotation during each day *with respect to the star* (or its parametric angle), but $1\frac{1}{4}$ rotations *with respect to the center of the galaxy* (or the x axis). After a 4-day year, it will have made $4 / 1 + 1 = 5$ total rotations. ($4 / 1 = \text{Rotations} / \text{revolutions}$.) Similarly, if the planet rotates in a clockwise direction, it only makes $\frac{3}{4}$ rotation *with respect to the center of the galaxy* (x axis) daily, so its total number of rotations per year is $4 / 1 - 1 = 3$.



Plotting the position of the generating point of a roulette uses the circle's rotation *with respect to the x axis*. When the terms from the previous parametric equations $(\mathbf{R} \pm \mathbf{r}) / \mathbf{r}$ are transformed to the equivalent $(\mathbf{R} / \mathbf{r}) \pm 1$, the rotation of the circle can be seen to correspond to the analogy of planetary rotation *with respect to the center of the galaxy* (x axis) described above. As noted previously for roulettes, the **Rotation** / **revolution** ratio also is the circumference ratio



There are numerous special cases of roulettes. Epicycloids where $\mathbf{R} = \mathbf{r}$ are cardioids, as seen in the Mandelbrot set, and those where $\mathbf{R} = 2\mathbf{r}$ are nephroids. Hypocycloids where $\mathbf{R} = 3\mathbf{r}$ are deltoids, and those where $\mathbf{R} = 4\mathbf{r}$ are astroids (popular for logos, including those of the Pittsburgh Steelers and the Portland, Oregon flag). Hypotrochoids where $\mathbf{R} = 2\mathbf{r}$ are ellipses; when $\mathbf{d} = 0$ the curve is a circle and when $\mathbf{d} = \mathbf{r}$ the curve is a line segment. Any epi- or hypotrochoid in which $\mathbf{d} = 0$ is a circle. There also are numerous interesting non-trochoidal roulettes, such as the cissoid of Diocles, generated by one parabola rolling around another.



Rhodonea, or rose curves, are remotely related to cycloids, although they are generated by a ray from the center of a circle which can be described in polar coordinates as

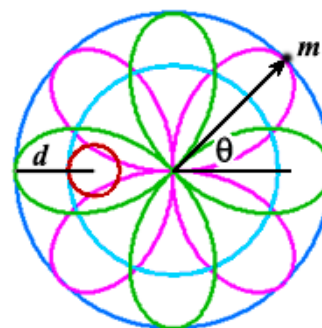
$$m = R \cos(k\theta) \quad \text{or} \quad m = R \sin(k\theta)$$

which leads to the parametric equations:

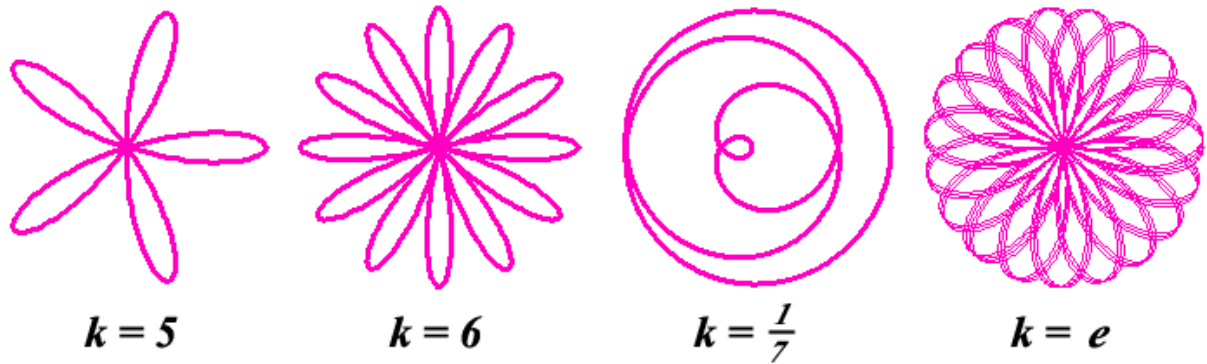
$$x = R \cos(k\theta) \cos(\theta)$$

$$y = R \cos(k\theta) \sin(\theta)$$

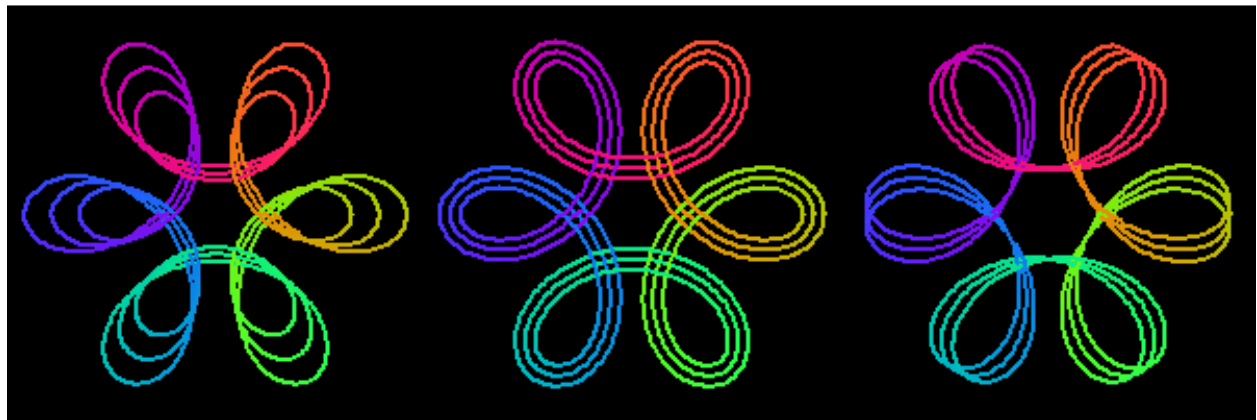
The image at the right shows that a **rhodonea** (rotated 45° by using sines rather than cosines) with $\mathbf{R} = 6$ and $\mathbf{k} = 2$ is identical to a **hypotrochoid** with $\mathbf{R} = 4$, $\mathbf{r} = 1$ and $\mathbf{d} = 3$.



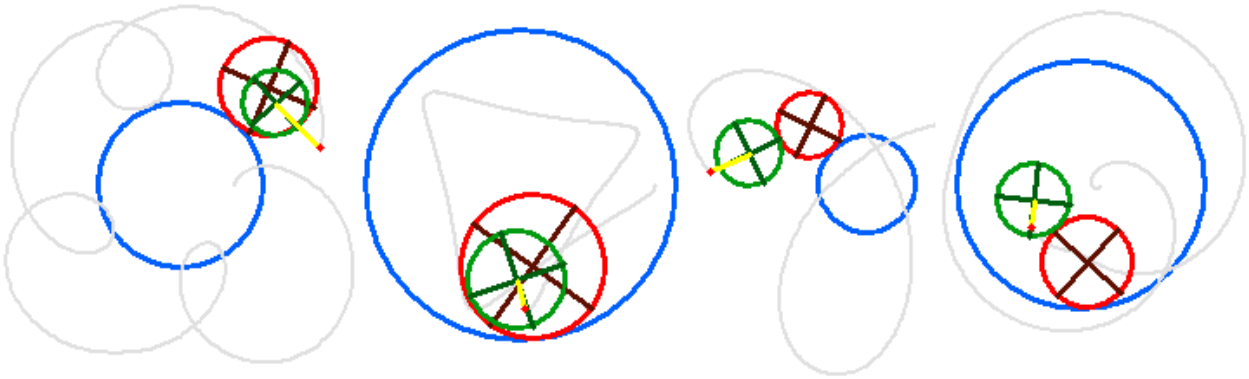
If k is an integer, the rhodonea has k or $2k$ petals, depending on whether k is odd or even (if odd, the curve traces over itself, so only k petals are seen). If k is rational ($k = \frac{kn}{kd}$ where kn and kd are integers) and kn and kd are reduced to lowest terms, then if both kn and kd are odd, the period of the curve is πkd ; otherwise the period is $2\pi kd$. If k is rational, but not an integer, the shape of the curve is more complex. If k is irrational, there are an infinite number of petals.



See **Appendix 1** at the end of this article for a table of representative rhodonea curves.



These hypotrochoid sets, all with the same values of R and r , show 3 ways to generate sets of related curves. The curves on the left all have the same value of d , but they are rendered using different scaling factors, producing “similar” curves with considerable overlapping. The curves in the middle are rendered using different values of d , producing dissimilar parallel curves. The curves on the right have the same value of d and are rendered with the same scaling factor. They are identical, but each one starts at a different offset value of θ .



These diagrams show that it is possible to generate curves using more than 2 circles. Both the 2nd and 3rd circles can be either epitrochoidal or hypotrochoidal, so there are 4 possible combinations. In the material world of plastic, paper and pens, if the 3rd circle is epitrochoidal, it will crash repeatedly into the fixed circle. Of course, this is not a problem in the virtual world. The QS Roulette 3 program uses the following formulae to link the 2nd and 3rd circles. But also in the virtual world, the 3rd circle can rotate in either direction and at any arbitrary speed, so the possible variations are endless.

$$x = F \cos \theta \pm G \cos \gamma \pm d \cos \delta$$

$$y = F \sin \theta \pm G \sin \gamma \pm d \cos \delta$$

2nd circle:

$$\text{epitrochoidal: } F = (R + r1) \quad \gamma = -(R/r1 + 1) * \theta$$

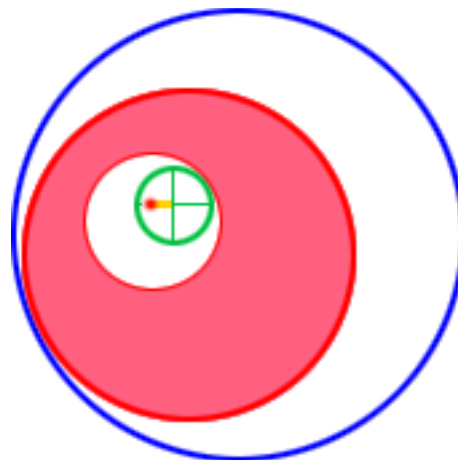
$$\text{hypotrochoidal: } F = (R - r1) \quad \gamma = (R/r1 - 1) * \theta$$

3rd circle:

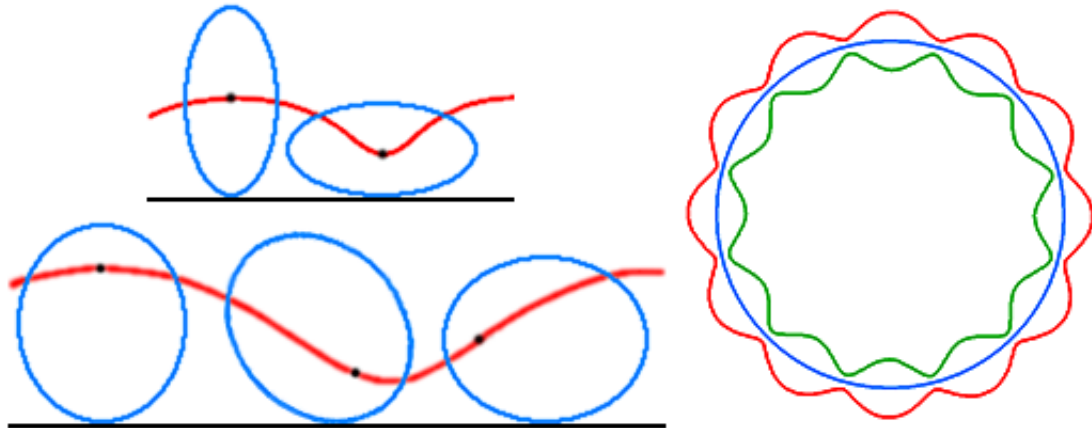
$$\text{epitrochoidal: } G = (r1 + r2) \quad \delta = -\text{ceiling}(r1/r2) * \gamma$$

$$\text{hypotrochoidal: } G = (r1 - r2) \quad \delta = \text{ceiling}(r1/r2) * \gamma$$

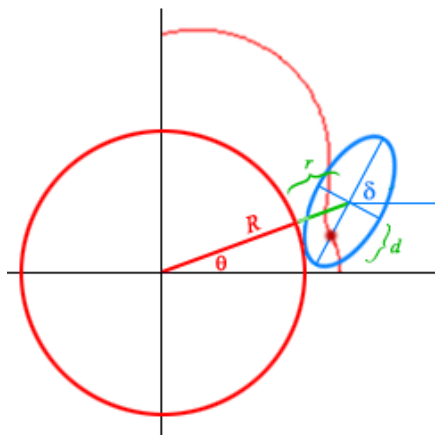
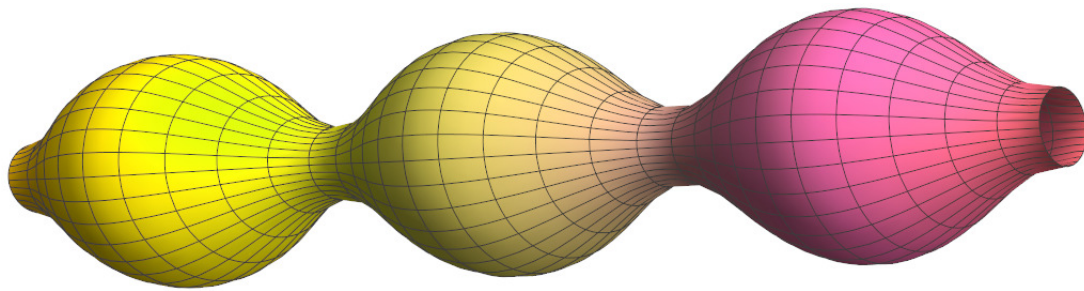
For the 2nd circle, the Spirograph typically uses a ring with as narrow a width as possible, which still can accommodate gear teeth on both sides without breaking. But the ring can be distorted, as shown on the right, with the 3rd circle revolving and rotating inside the cut-out circle in hypotrochoidal fashion.



Finally, it is possible to program the moving curve as a non-circular shape, such as an ellipse, or even a distorted shape like some of the ones found in Spirographs. **Sturm's roulette** is traced by the center of an ellipse rolling on a line, and **Delaunay's roulette** is traced by one of the ellipse's focal points. These curves can be adapted to follow a circle rather than a line.



Parenthetically, Delaunay's roulette also is known as an **elliptic catenary**, and its surface of revolution is called an **unduloid**, which has a constant mean curvature.

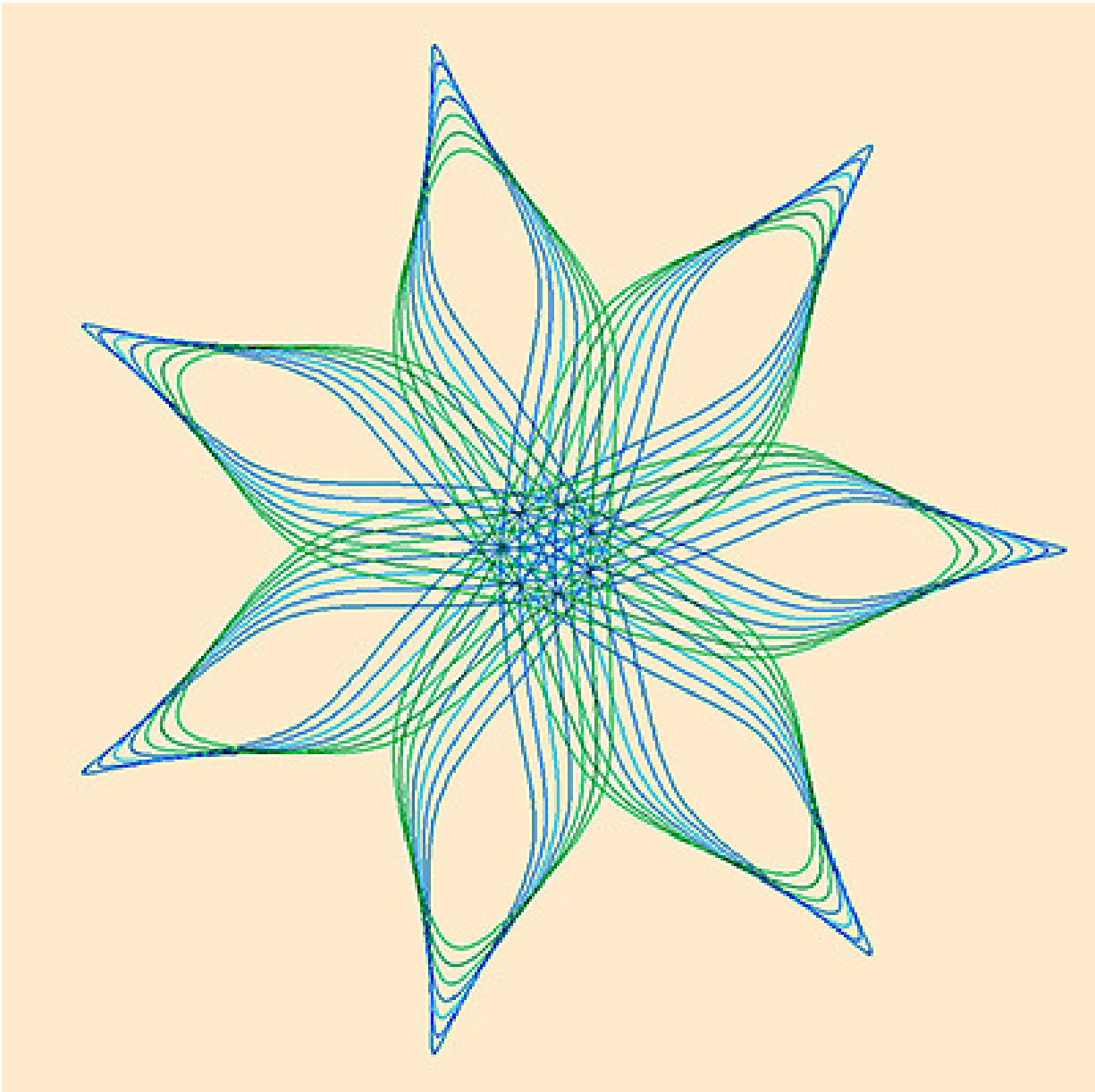


To plot an elliptical roulette, just as with a circular roulette, 2 determinations must be made:

1) the angle of rotation δ (with respect to the x axis) of the ellipse; and 2) the distance $R \pm r$ from the center of the ellipse to the center of the circle.

For details about how this can be done, see **Appendix 2** at the end of this article.

Many variations on the elliptical theme can be used to simulate non-circular moving objects which produce interesting images. This quasi-elliptical hypotrochoid image has an $R : r$ ratio of $7 : 4$. The ratio of the minor to major axes of the “ellipse” is $3 : 4$. The value of d is varied for each curve according to an elliptical formula, instead of the value of r . Thus, the moving object is a circle whose generating point changes in an elliptical fashion. The image is similar to one which is demonstrated in a video (<https://www.youtube.com/watch?v=eC9dGWEc10Q>) showcasing a new version of the Spirograph called Wild Gears (<http://www.wildgears.com>). For that image, a triangular moving piece with rounded vertices was used.

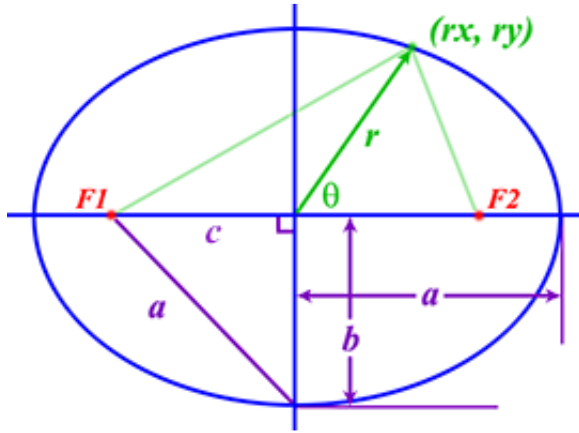


Appendix 1: Rhodonea Curves

These are the unique rhodonea curves for **kn** and **kd** ranging from 1 to 10.

$n \backslash d$	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

Appendix 2: Plotting Elliptical Roulettes



Ellipses are characterized by major and minor axes, of length $2a$ and $2b$. Two foci, **F1** and **F2**, are located symmetrically on the major axis. The sum of the distances

from any point **r** of an ellipse to both foci is constant and equals $2a$. The focal distance is:

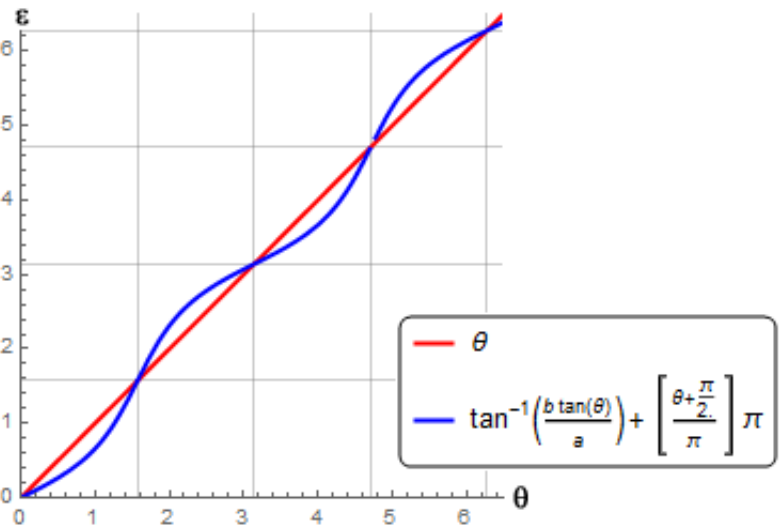
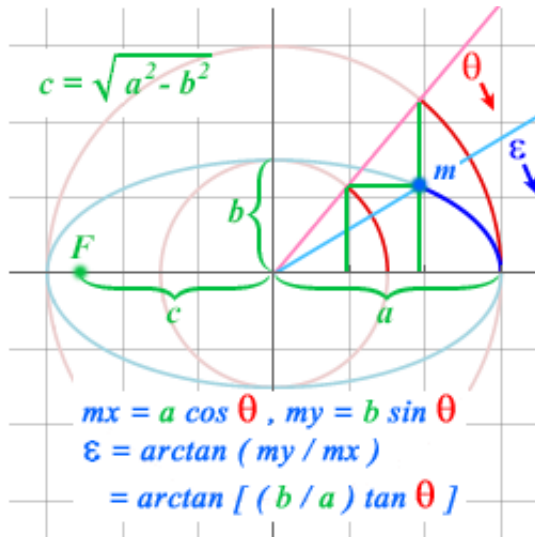
$$c = \sqrt{a^2 - b^2}$$

The Cartesian formula for an ellipse is:

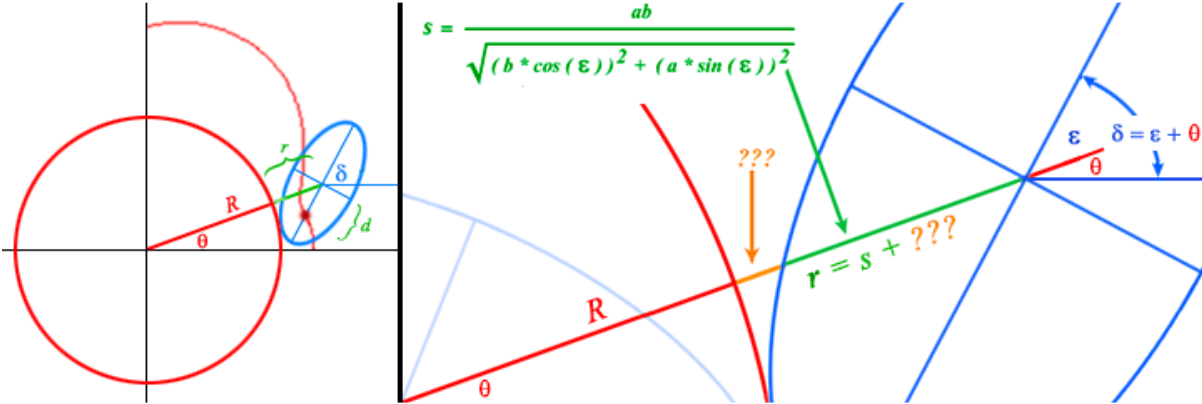
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

r can be described as:

$$r(\theta) = \frac{ab}{\sqrt{(b \cos \theta)^2 + (a \sin \theta)^2}}$$



This is a parametric representation of an ellipse. The point **m** is calculated from the **parametric angle** θ . The graph shows that the **elliptical angle** ϵ is not the same as θ except at multiples of $\pi / 2$. This relationship can be used to calculate the angles of rotation in an elliptical roulette. (The brackets in the term with various instances of π represent a floor $\lfloor \cdot \rfloor$ function; this term is necessary to keep the graph rising instead of dropping back down at intervals of π .)



To plot an elliptical roulette, 2 determinations must be made: 1) the angle of rotation δ (with respect to the x axis) of the ellipse; and 2) the distance $R \pm r$ from the center of the ellipse to the center of the circle.

1) As an **ellipse** (or other curve) *revolves* around a **fixed circle**, its center describes an angle θ with the x axis. Meanwhile, each *rotation* of the ellipse describes an angle ϵ which reaches a multiple of 2π radians each time a full rotation (with respect to θ , not the x axis) is completed. Calculating the rate of rotation of the ellipse requires comparing its circumference to that of the fixed circle, but there is no basic algebraic solution for calculating the circumference. A complete elliptical integral of the second kind is required:

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta$$

where e is the eccentricity

$$e = c/a$$

This gives the infinite series:

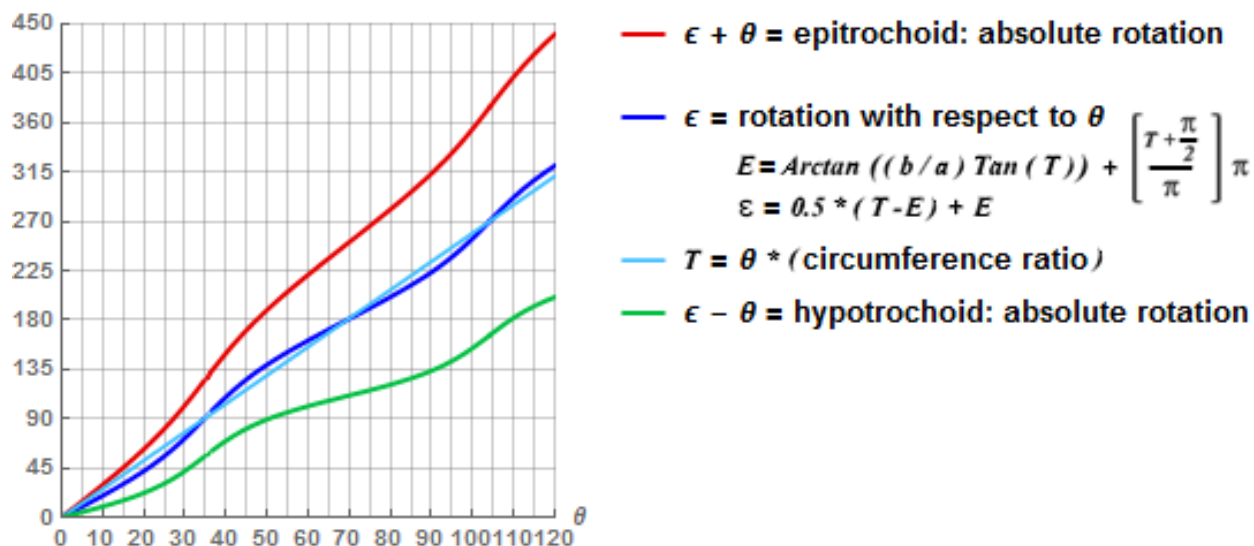
$$C = 2\pi a \left[1 - \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{2^n n!} \right)^2 \frac{e^{2n}}{2n-1} \right]$$

For the purpose of recreational computer graphics programming, either of two approximations by S. Ramanujan is more than precise enough:

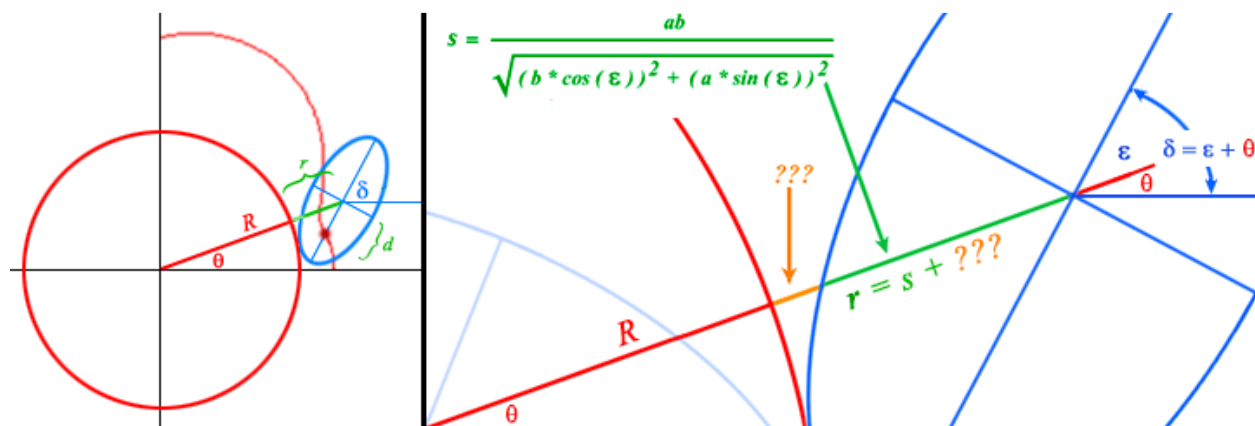
$$C \approx \pi(a+b) \left(1 + \frac{3h}{10 + \sqrt{4-3h}} \right), \text{ where } h = (a-b)^2/(a+b)^2$$

$$C \approx \pi \left[3(a+b) - \sqrt{(3a+b)(a+3b)} \right] = \pi \left[3(a+b) - \sqrt{10ab + 3(a^2 + b^2)} \right]$$

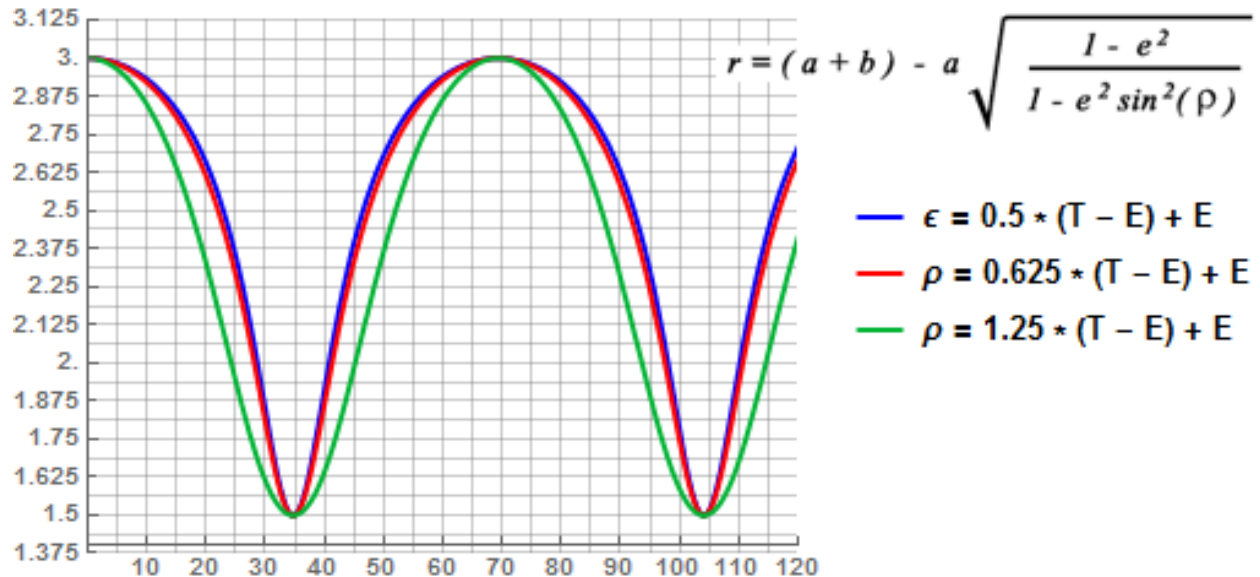
Once the circumference is calculated, the revolution angle θ is multiplied by the ratio of circumferences of the **circle** to the **ellipse** to yield the angle labeled **T** in the graph below. However, the *rate* of rotation of the **ellipse** is not constant because its varying curvature. The above elliptical angle formula is used to calculate a preliminary angle **E**, which then is adjusted halfway to **T** to yield the rotational angle ϵ (with respect to θ). Finally, θ either is added to ϵ (for epitrochoids) or subtracted from ϵ (for hypotrochoids) to obtain the rotational angle δ (with respect to the x axis) which is used for plotting the roulette.



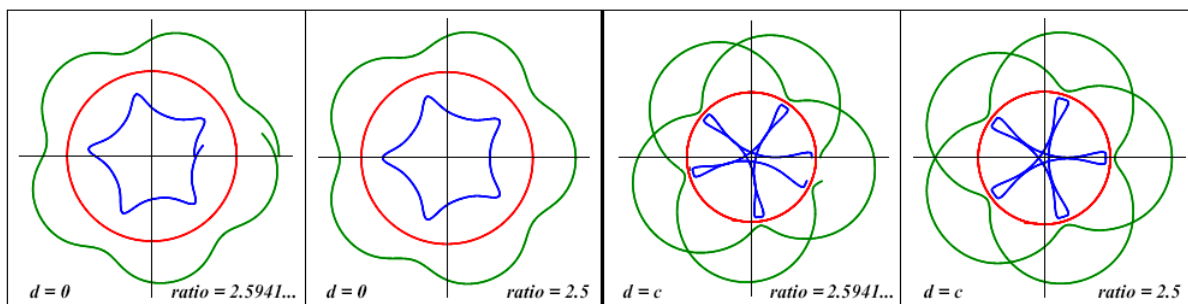
2) As with circular roulettes, the distance from the center of the ellipse to the center of the circle is **R**, the constant radius of the fixed circle, plus or minus **r**, which in this case is harder to determine. As this diagram shows, **r** is composed of a segment labeled **s**, which can be calculated readily, and another segment, labeled **???**, which cannot, unless it equals 0, which only happens at multiples of $\pi / 2$ when the segment goes through the tangent point of the ellipse and circle.



A reasonable approximation can be made using Sturm's roulette curves with empirically-chosen adjustments (“ e ” = eccentricity = c / a) :



It is difficult to match **R**, **a** and **b** so that the circumference ratio is a relatively simple rational number that results in a roulette that returns to its starting point in a small number of revolutions. For example, when **R** = **4b** and **a** = **2b**, the ratio is 2.594100831843412... and it would take 124,620,516 revolutions for the roulette curve to close. Artificially rounding the ratio can result in simple and symmetrical curves. This is analogous to manipulating the number of gear teeth in a Spirograph. In this diagram, the actual ratio is rounded to the nearest 0.25 for roulettes in which **d** = 0 and **d** = **c** (the focal distance) respectively.



The elliptical formulae can be used in different ways, often easier to program, to produce interesting images. For example, the moving object can be treated as a circle whose radius **r** varies between **a** and **b**. The circumference ratio is set to **R / a** and is used to calculate the rotation angle **ε** of the moving circle. The parametric equations described above are used to calculate **r** as follows:

$$mx = a \cos \varepsilon \quad my = b \sin \varepsilon \quad r = \sqrt{mx^2 + my^2}$$

Another method, in the name of producing aesthetically pleasing images while abandoning attempts at mathematical accuracy, keeps the moving circle unchanged but varies **d**; the image on page 7 was rendered using this technique.

$$d = \frac{ab}{\sqrt{(b \cos \varepsilon)^2 + (a \sin \varepsilon)^2}}$$

All images were rendered with software programmed by the author, with the exception of the linear trochoid, Sturm's and Delaunay's roulettes, 3-circle Spirograph illustration and elliptical calculation diagrams, which were made in PhotoShop; the unduloid and the graphs, which were rendered in Mathematica; and the Steelers and Portland flag logos, which are found in countless locations across the World Wide Web.